IP 01. Erdős-Szekeres type problems for colored point sets and compatible graphs

In 1935 Erdős and Szekeres [8] considered a problem about the existence of a number \( g(k) \) such that any set \( S \) of \( g(k) \) points in general position (no three points of \( S \) lie on a line) in the plane has a subset of \( k \) points that are the vertices of a convex \( k \)-gon. Later Erdős and Guy stated the following more general question: “What is the least number of convex \( k \)-gons determined by any set of \( n \) points in the plane?”. Both versions turned out to be rather challenging and have attracted many researchers. Meanwhile, there exists a whole family of problems based on these questions, see e.g. [1], Chapter 8 of [6], and IP06 for more details.

We focus on variations of Erdős-Szekeres type problems, where the points of a given set \( S \) belong to different classes – usually described as “colors”. This colorful family of problems was introduced in 2003 by Devillers et al. [7]. Recent progress on this and other Erdős-Szekeres type problems has been made by the PIs Aichholzer, Hurtado, Pach, Valtr, and Welzl, so this is certainly a topic which will profit from the joint research effort within this CRP.

We plan to investigate the following specific problems:

**Empty monochromatic convex quadrilaterals**: Let \( S \) be a bichromatic set of \( n \) points in the plane, some of them colored red, and the rest colored blue. A quadrilateral is called monochromatic if it is spanned by four points of the same color, and empty if no point (of any color) lies in its interior. Does there exist an empty monochromatic convex quadrilateral on every sufficiently large point set \( S \) in the plane?

This problem has been explicitly mentioned in Section (iv) of the Call for Outline Proposals. Recently [4] we could prove that in every sufficiently large bichromatic point set in the plane there exists a not necessarily convex monochromatic 4-hole.

**Empty monochromatic simplices**: Let \( S \) be a \( c \)-colored set of \( n \) points in \( d \)-space. For \( d = 2 \) and \( c = 2 \) the task is to prove or disprove the conjecture that any bi-colored set of points in the plane determines a quadratic number of empty monochromatic triangles. A first super-linear bound of \( \Omega(n^{5/4}) \) was shown in [2], which was consequently refined by Pach and Tóth [9] to the currently best known bound of \( \Omega(n^{4/3}) \). A joint effort of these research groups might help to settle the problem. For \( d \geq 3 \) and \( c \geq 2 \) we aim to generalize these results to get (better) bounds for monochromatic empty simplices in multi-colored point sets in general dimension.

**Blocking Delaunay triangulations**: Let \( B \) be a set of blue points in general position in the plane. With \( DT(B) \) we denote the Delaunay triangulation of \( B \). Adding a set \( R \) of red points, we get a bichromatic point set \( B \cup R \). We call an edge blue, red, or black if it is spanned by two points of \( B \), two points of \( R \) or one point of each \( B \) and \( R \), respectively. We are interested in the minimum size of \( R \), such that for each set \( B \), the Delaunay triangulation \( DT(B \cup R) \) (of the bichromatic set \( B \cup R \)) contains no blue edge. This problem belongs to the class of proximity graphs on (uncolored) point sets \( S \) and can be seen as a negative version of witness graphs, the so called witness Delaunay graphs, see IP04 for details. The problem also constitutes a variant of the obstacle representation graph, where the obstacles are points (see IP05 for a description and reference [16] given there). Thus we will investigate this question in collaboration with IP04 and IP05. Moreover, an inter-CRP cooperation with PIs from the EuroGiga CRP Voronoi is planned, as the dual of the Delaunay triangulation is the Voronoi diagram.
The second class of problems we will investigate are compatible geometric graphs. Here compatible comes in two different flavors. A plane perfect matching on a set $S$ of an even number of points in the plane is a crossing-free geometric graph, where every point in $S$ has degree exactly 1. Two plane geometric graphs are compatible if they both exist on the same point set and their union is plane. They are called disjoint if the two graphs have no edge in common.

We will investigate the “Compatible Matching Conjecture”: For every perfect matching on a set $S$ of $4n$ points in the plane, there exists a disjoint compatible perfect matching.

Related questions exist on (disjoint) compatible spanning paths/trees/circles and also on (disjoint) compatible (pointed) pseudo-triangulations, including the problem of the length of transformations (successively compatible sequences). Being of interest themselves, results for these problems may also give additional insight for the original question on (disjoint) compatible perfect matchings. Moreover, these questions are related to the number of such structures, and thus a joint effort with IP07 (enumeration tasks) to understand the structure behind these graphs seems promising.

The second flavor of compatible graphs comes as a geometric variant of graph isomorphism. For a set $S$ of points in the plane, let $|S|$ denote its cardinality and $|\text{CH}(S)|$ the cardinality of its convex hull. We consider the “Compatible Triangulation Conjecture”: Let $S_1$ and $S_2$ be two sets of points in the plane in general position and without fixed correspondence. Then compatible triangulations on $S_1$ and $S_2$ exist if the following holds: (1) $|S_1| = |S_2|$, (2) $|\text{CH}(S_1)| = |\text{CH}(S_2)|$. (Note that in this context the terms “compatible” and “isomorphic” are used synonymously in the literature, though there are some subtle differences [3].)

Isomorphic triangulations play a major role in morphing, used in research areas like computer graphics, animation, and modeling. There, also isomorphic triangulations of two simple polygons are sometimes considered, for instance for the construction of so called sweep volumes. In general, isomorphic triangulations need not exist for a pair of polygons. Thus additional points, so called Steiner points, are added to the polygons, see for instance [5,10].

Progress on this problem has been obtained in partial collaboration with the Spanish team (IP04), see [3]. The above conjecture might even hold if the correspondence between the extreme points of $S_1$ and $S_2$ is given by a cyclically shifted labeling. Solving these conjectures, as well as extending them to related structures, e.g. pseudo-triangulations, is a challenging task which will require the joint effort of this CRP.

Bibliography.


