IP 03. Arrangements and Higher Dimensions

This project assembles three research directions each lead by a senior researcher. The areas and the responsible researchers are:

- Arrangements of Pseudolines, Stefan Felsner (Technical University)
- Enumeration of Polycubes, Günter Rote (Free University)
- Tverberg Type Theorems, Günter M. Ziegler (Technical University)

Following the instructions for preparing the proposal the CVs of Günter Rote\textsuperscript{1} and Günter M. Ziegler\textsuperscript{2} have not been made part of the proposal, they are available from the web.

Arrangements of Pseudolines  Pseudoline arrangements generalize arrangements of straight lines in a very natural way. Elementary and intuitive in nature they serve as a concrete model for oriented matroids of rank 3. For background information see [15, 17, 11]. We are interested in combinatorial properties of such arrangements. With an arrangement of \( n \) lines or pseudolines we consider the combinatorial data \( t_i = \# \) of intersection points of \( i \) lines and \( p_k = \# \) of faces bounded by \( k \) lines. Bounds for and relations between these numbers have been studied since Levi introduced pseudolines in 1926. Despite the steady progress in the field there remain many open problems and conjectures [9]. We mention two of the most intriguing:

- Show that every arrangement of \( n > 13 \) pseudolines has at least \( n/2 \) ordinary points, i.e., \( t_2 \geq n/2 \).
- Hirzebruch [6] proved \( t_2 + \frac{3}{2}t_3 \geq n + t_5 + 3t_6 + 5t_7 + \ldots \), when \( t_n = t_{n-1} = t_{n-2} = 0 \) for arrangements of lines. Find an elementary proof of this inequality that also applies to arrangements of pseudolines.

We plan to tackle these and related problems by means of zonotopal tilings associated to arrangements. A particularly interesting aspect of the general question is that there are \((t_i)_{i \geq 2}, (p_k)_{k \geq 3}\) that can be realized by an arrangement of pseudolines but not by an arrangement of lines, see [12]. We are specially keen to identify further examples of this phenomenon.

In the spirit of red-blue problems for point sets it is natural to consider colored arrangements of pseudolines and ask for bounds and relations among the refined data \( t_{i,j} = \# \) intersection points of \( i \) red and \( j \) blue lines and the correspondingly defined \( p_{k,l} \).

We also want to look at more general structures than arrangements of pseudolines. Interesting examples range from simple families of 2-crossing \( x \)-monotone curves to families of pairwise intersecting closed curves. It may be worth looking at families of 2-crossing curves that do not induce a bounded face of degree \( \geq 5 \), for arrangements of pseudolines this condition enforces a very strong and well understood structure. An important question about the second class was asked by Richter and Thomassen:

- Is it true that a simple family of \( n \) pairwise intersecting closed curves has at least \((1 - o(1))n^2\) crossings?

Another important problem about arrangements of pseudolines is to give good bounds on their number. Progress on this problem is related to combinatorial encodings of

\textsuperscript{1} http://page.mi.fu-berlin.de/rote/about_me/CV-short-2010.pdf
\textsuperscript{2} http://www.math.tu-berlin.de/~ziegler/presse.shtml#CV
arrangements and can lead to insights that are helpful for other questions. E.g. the recent improvements of lower and upper bounds by Felsner and Valtr [13] inspired the disprove of a conjecture made independently by Knuth and in the context of decision theory by Fishburn. Presently, the base 2 logarithm of the best bounds is $0.188n^2$ (lower) and $0.66n^2$ (upper). To improve the bounds we will study conditions and obstructions for the combination of smaller arrangements into bigger ones. This approach is closely related to questions about colored arrangements.

**Enumeration of Polycubes** A polyomino is a connected subset of squares of the square grid in the plane. Polyominoes and their higher-dimensional generalizations, polycubes, have been a fascinating subject, starting from recreational mathematics and ranging to fundamental questions in statistical physics (where they are called *lattice animals*).

In particular, the question of finding or estimating the number $A_d(n)$ of $d$-dimensional polycubes with a given number $n$ of cells have challenged mathematicians, computer scientists [16], and theoretical physicists [10] alike. Research in this area is characterized by an interplay between computer methods on the one hand and analytical and theoretical methods on the other hand. For example, we could obtain the currently best lower bound on Klarner’s constant $\lambda_2 = \lim_{n \to \infty} \sqrt[n]{A_2(n)}$ by using a novel setup and extensive computer calculations [4]. In a different work [5], we have also considered the limit $d \to \infty$, which leads to asymptotic expansions of the “free energy” that are of interest in theoretical physics [14], and we could prove some expansions that had hitherto been derived only by non-rigorous methods.

In particular we consider the polycubes which $n$ cubes that span $n - d$ dimensions, for fixed (small) $d$ and for variable $n$. Explicit formulas for $d = 1$ and $d = 2$ are known [5]. We have also derived a formula for $d = 3$ (announced in [5]). It is based on a graph-theoretic model for polycubes, an exclusion-inclusion formula using various types of graph and tree “patterns” that can occur in polycubes and elaborate case distinctions. Going to higher values of $d$ would provide more terms in the asymptotic expansions of the free energy, with rigorous proofs. However, carrying the current approach beyond $d = 3$ would create case distinctions that surpass the patience of humans. The goal of the project is create a theoretical setup that would allow the case distinctions to be made by a computer. One core problem is the enumeration of small graphs and the identification of isomorphic copies of (small) subgraphs in other graphs.

There is also a conjecture about the general form of the formula, for varying $d$, that would follow from a such a general setup.

**Tverberg Type Theorems** Discrete-geometric partition theorems such as Radon’s theorem and its generalizations are among the most fundamental results about finite sets of points and convexity. For some of the colored and topological versions the only known proofs involve applications of algebraic topology. Our research in this area uses our new 2009 Colored Tverberg theorem as a starting point for diverse investigations. We concentrate on the 2-dimensional case, which (a) shows the full complexity of the general case, (b) connects to geometric graph theory, and (c) for small parameters is accessible to enumeration approaches. We will in particular study the number of solutions, and try to develop algorithms for finding solutions.

Motivated by basic problems in Computational Geometry such as the $k$-set problem Bárány, Füredi & Lovász [2] asked for “a colored version of Tverberg’s theorem”. Such
A theorem was first achieved in 1992 by Živaljević & Vrećica [20]. However, their Colored Tverberg Theorem is not tight, as one would want to have the same conclusion assuming with a weaker assumption $|C_i| \geq r$ instead of $|C_i| \geq 2r - 1$. Moreover, the “configuration space/test map” method yielded the result only for primes $r$. And the result does not have the classical 1966 Tverberg Theorem as a special case. Thus the following result, achieved in the fall of 2009 by Blagojević, Matschke & Ziegler [8], is a considerable breakthrough.

**New Colored Tverberg Theorem.** Let $d \geq 1$ and let $r \geq 2$ be a prime. Given any $m + 1$ point sets (“color classes”) $C_0, C_1, \ldots, C_m \subset \mathbb{R}^d$ of size $|C_i| \leq r - 1$, with $|C_0| + \cdots + |C_m| = (d+1)(r-1) + 1$, one can find $r$ disjoint sets $A_1, \ldots, A_r \subset C_0 \cup \cdots \cup C_m$ with $|A_i \cap C_j| \leq 1$ (“rainbow simplices”) such that $\text{conv}(A_1) \cap \cdots \cap \text{conv}(A_r) \neq \emptyset$.

The New Colored Tverberg Theorem was proved by advanced algebraic topology methods (see also [7] [19] for a simpler, degree-theoretic proof.) In particular these methods also imply a topological version, which in the special case $|C_i| = 1$ reduces to the “topological Tverberg theorem” of Bárány, Shlosman & Szücs [3]. A drawback is that the proof of [8] again only works when $p$ is a prime. A simple combinatorial proof is only known for the case $d = 1$.

We propose to study the following problems:

- There should be an elementary/combinatorial proof the New Colored Tverberg Theorem, at least for $d = 2$, which would be valid for all $r \geq 2$.
- Are there always many (colored) Tverberg partitions? For the classical (not colored) version of Tverberg’s theorem, Sierksma’s conjecture predicts that there should always be at least $(r - 1)!^d$ distinct partitions. We hope that the New Colored Tverberg Theorem will help with the counting problem.
- On the other hand, we hope that enumeration methods – including the techniques and the data bases of Aichholzer et al. – could help to explore and identify critical cases, such as point configurations with relatively few colored Tverberg partitions, which could yield insight (and eventually counter-examples).
- Tverberg’s Theorem, and all its colored versions, are pure existence theorems for certain types of partitions. However, it has never been shown that any of the Tverberg-type partitions are hard to find (in a complexity-theoretic sense). We want to explore this — starting with the probably much simpler setting of Bárány’s 1982 “Colorful Caratheodory Lemma” [1].

**Bibliography.**


